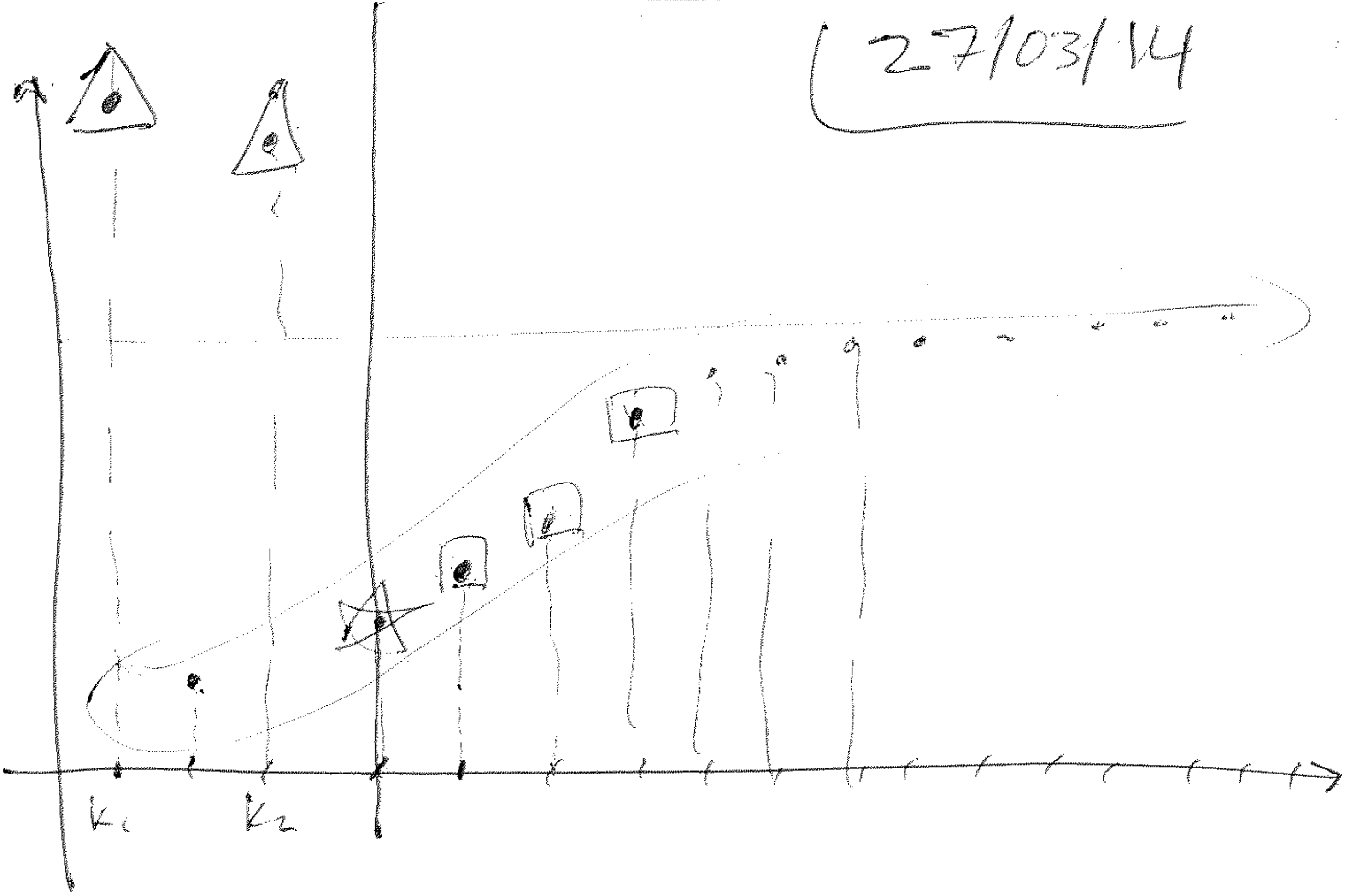
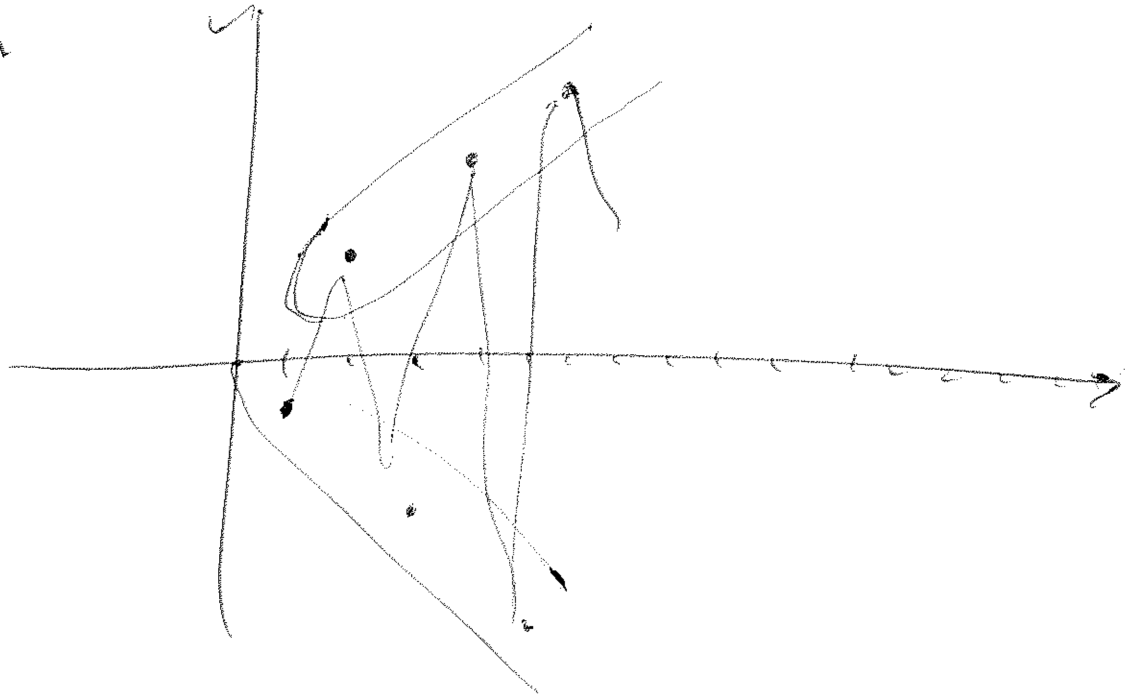
27/03/14



Take $n_1 = k_m + 1$. Now n_1 is <u>not</u> a peak point. Hence there exists $n_2 > n_1$ s.t. $a_{n_2} \geq a_{n_1}$. Now, n_2 is not a peak point. Hence there is $n_3 > n_2$ s.t. $a_{n_3} \geq a_{n_2}$. Now, n_3 is not peak; there is $n_4 > n_3$ s.t. $a_{n_4} \geq a_{n_3}$, etc.

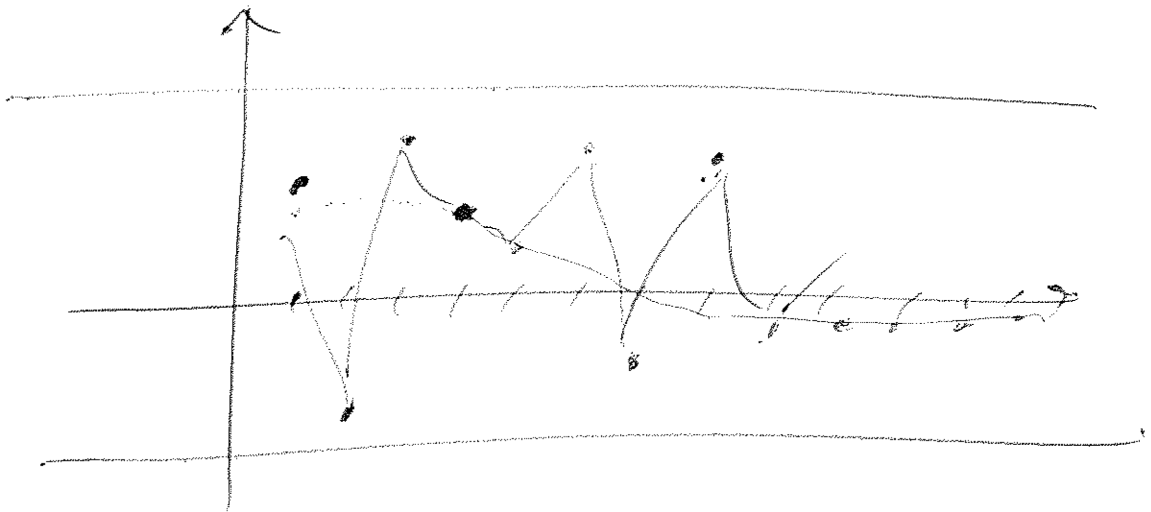
Thus, we obtain $(a_{n_k})_{k=1}^{\infty}$, which is monotone increasing. $\square$

Ex. $(-1)^n n$



Theorem. (Bolzano - Weierstrass).
Every bounded sequence has a convergent subsequence.

Proof



Every seq. has a ~~conv~~ monotone subseq. by lemma. This subseq. must be bounded and hence convergent. $\square$

Theorem. If $(a_n)_{n=1}^{\infty}$ is Cauchy, then $(a_n)_{n=1}^{\infty}$ converges.

Proof. Step 1. $(a_n)_{n=1}^{\infty}$ is bounded.

In the def of Cauchy, take $\varepsilon = 1$. There exists N s.t. $k, n \geq N \Rightarrow$

$|a_k - a_n| < 1$. Then, in particular,

$|a_k - a_N| < 1$. Then

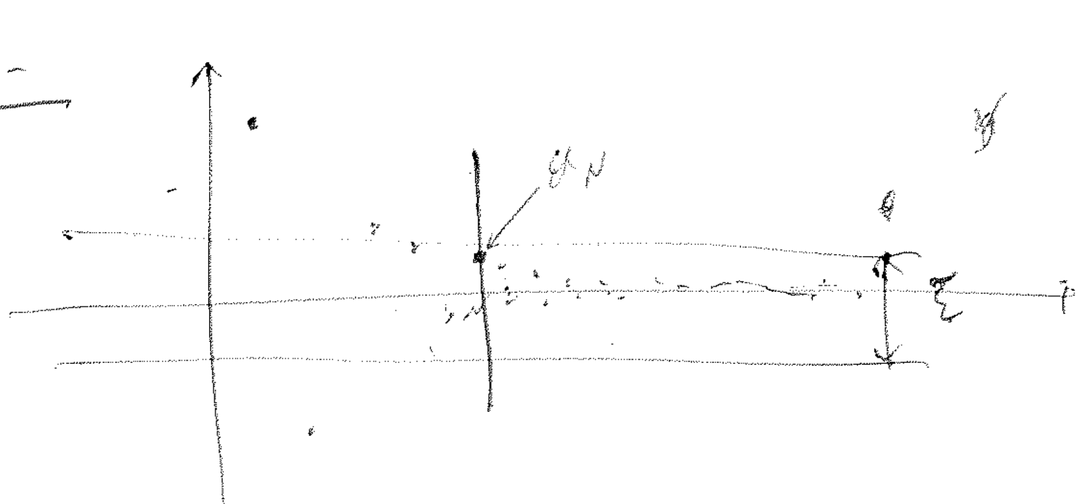
$|a_k| - |a_N| < 1$, which means

$|a_k| < 1 + |a_N|$ $(k \geq N)$

Take $M = \max\{a_1, a_2, \dots, a_{N-1}, |a_N| + 1\}$.

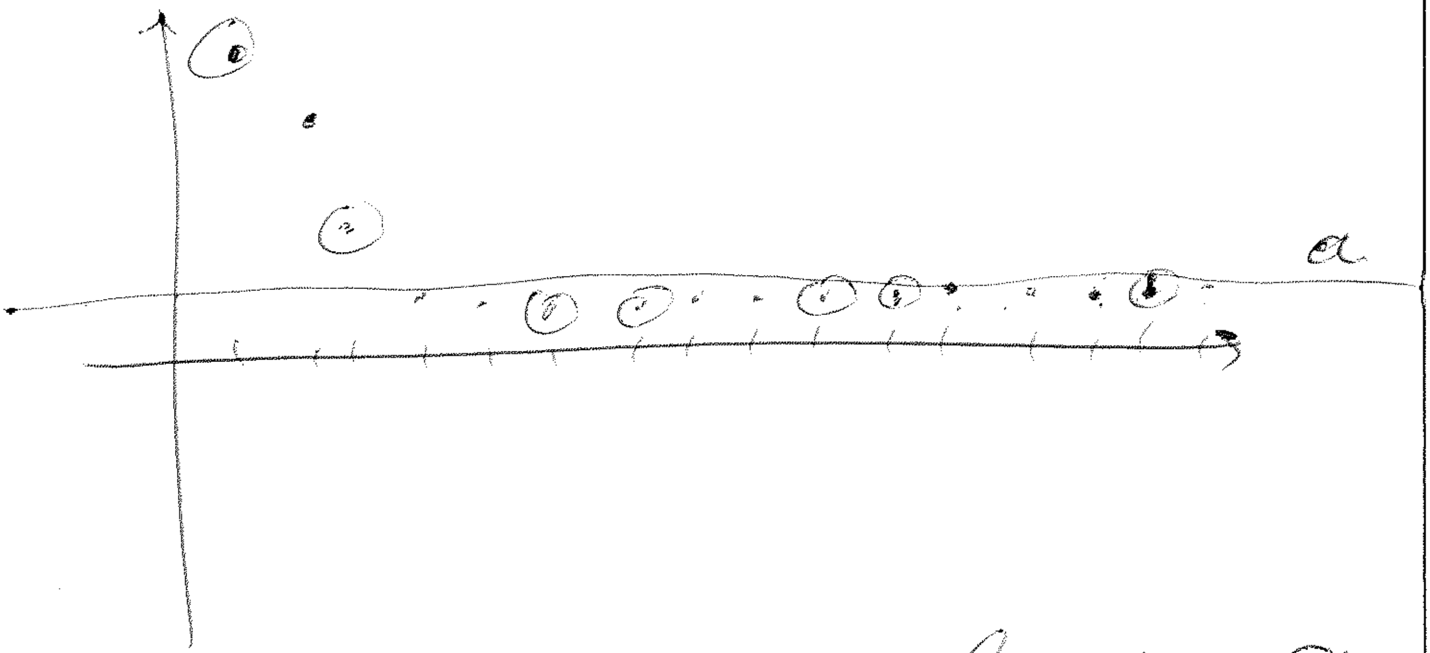
Obviously, $|a_k| \leq M$ for all $k \in \mathbb{N}$.

Step 2.



By the B-W theorem, $(a_n)_{n=1}^{\infty}$ has a convergent subseq. $(a_{n_k})_{k=1}^{\infty}$. Assume $\lim_{k\to\infty} a_{n_k} = a$.

We prove that $\lim_{n\to\infty} a_n = a$.



Fix $\varepsilon > 0$. Because $\lim_{k\to\infty} a_{n_k} = a$, there is $N_1 \in \mathbb{N}$ s.t. $k \geq N_1$ $\Rightarrow |a_{n_k} - a| < \varepsilon/2$.

Because $(a_n)_{n=1}^{\infty}$ is Cauchy, there is $N_2 \in \mathbb{N}$ s.t. $n, k \geq N_2$

$\Rightarrow |a_n - a_k| < \frac{\varepsilon}{2}$.

Consider, assuming $n \geq \max\{N_1, N_2\}$,

$|a_n - a| = |a_n - a_{n_k} + a_{n_k} - a| \leq$

(here, k is big enough to ensure $n_k \geq \max\{N_1, N_2\}$)

$\leq |a_n - a_{n_k}| + |a_{n_k} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$= \varepsilon$.

Thus, $|a_n - a| < \varepsilon$. $\square$

Lec 12.1

Series

Def. A series is a formal sum $\sum_{n=1}^{\infty} a_n$, where $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$.

Consider the sequence $(S_n)_{n=1}^{\infty}$, where $S_n = \sum_{k=1}^{n} a_k$ (partial sum of $\sum_{n=1}^{\infty} a_n$).

Def. $\sum_{n=1}^{\infty} a_n$ converges if $(S_n)_{n=1}^{\infty}$ converges.

$\sum_{n=1}^{\infty} a_n = a$ if $\lim_{n\to\infty} S_n = a$.

Theorem. If $\sum_{n=1}^{\infty} a_n$ converges,

the $\lim_{n\to\infty} a_n = 0$.

Proof. Note that

$$a_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k = S_n - S_{n-1}.$$

Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n$$
$$- \lim_{n\to\infty} S_{n-1} = a - a = 0.$$

$\square$

Remark. Converse not true.

Ex. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Proof later.

Ex. Does $\sum_{n=1}^{\infty} \underbrace{\frac{n+1}{n+2}}_{a_n}$ converge?

$\lim_{n\to\infty} a_n = 1 \neq 0$.

Theorem (Cauchy criterion).

$\sum_{n=1}^{\infty} a_n$ converges if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t.

$q > p \geq N \Rightarrow \left| \sum_{k=p+1}^{q} a_k \right| < \varepsilon.$

Proof. ~~A The more~~

The series converges $\Leftrightarrow (s_n)_{n=1}^{\infty}$ converges $\Leftrightarrow (s_n)_{n=1}^{\infty}$ is Cauchy

$\Leftrightarrow$ For every $\varepsilon > 0$, there is $N \in \mathbb{N}$ s.t.

$p, q \geq N \Rightarrow |s_q - s_p|$

$$= \left| \sum_{k=1}^{q} a_k - \sum_{k=1}^{p} a_k \right|$$

$$= \left| \sum_{k=p+1}^{q} a_k \right| < \varepsilon. \qquad \square$$

Example. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof. Assume this converges.

Apply Cauchy criterion with $\varepsilon = \frac{1}{3}$.

There exist $N \in \mathbb{N}$ s.t. $q > p \geq N$

$$\Rightarrow \left| \sum_{n=p+1}^{q} \frac{1}{n} \right| < \frac{1}{3}.$$

In particular, this must hold for $p = N$ and $q = 2N$.

Then

$$\left|\sum_{n=N+1}^{2N} \frac{1}{n}\right| = \left| \cancel{\frac{1}{N}} \frac{1}{N+1} + \frac{1}{N+2} + \dots + \frac{1}{2N}\right|$$

$$> \left|\underbrace{\frac{1}{2N} + \frac{1}{2N} + \dots \frac{1}{2N}}_{N \text{ times}}\right|$$

$$= \left|\frac{\cancel{N}}{2\cancel{N}}\right| = \frac{1}{2} > \frac{1}{3}.$$

Contradiction. $\square$

Convergence laws ~~from~~ for series

Assume $\sum_{n=1}^{\infty} a_n = a$ and $\sum_{n=1}^{\infty} b_n = b$ converge, $\lambda \in \mathbb{R}$. Then

$$\sum_{n=1}^{\infty} (a_n + b_n) = a + b, \quad \sum_{n=1}^{\infty} \lambda a_n = \lambda a.$$

This follows from the corresponding results for seqs; ~~also $\left(\sum_{n=1} a_n\right)$ $\sum a_n \sum b_n = ab$.~~

Also, $\cancel{\sum_{n=1}^{\infty} a_n b_n}$

$$\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right) = ab.$$

$$\sum_{n=1}^{\infty} a_n b_n \neq ab$$

Comparison theorem.

If $0 \le a_n \le b_n$, ~~then~~ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof. Apply comparison theorem for sequences to partial sums. $\square$

Examples. 1) $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p < 1$, diverges, because $\frac{1}{n^p} > \frac{1}{n}$.

2) $\sum_{n=1}^{\infty} \frac{1}{n+2^n}$ converges. Why?

$0 < \frac{1}{n+2^n} < \frac{1}{2^n}$, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges as a geometric series.

Please review
geometric series from 1051.
Convergence tests from 1051.

Lec. 13